CYLINDRICITY OF COMPLETE EUCLIDEAN SUBMANIFOLDS WITH RELATIVE NULLITY

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ABSTRACT. We show that a complete Euclidean submanifold with minimal index of relative nullity $\nu_0 > 0$ and Ricci curvature with a certain controlled decay must be a ν_0-cylinder. This is an extension of the classical Hartman cylindricity theorem.

1. INTRODUCTION

The simplest examples of isometric immersions $f : M^n \to \mathbb{R}^m$ such that the index of relative nullity is positive everywhere are the s-cylinders. The isometric immersion f is said to be an *s-cylinder* if there exists a Riemannian manifold N^{n-s} such that M^n, $\mathbb{R}^m$ and f have factorizations

$$M^n = \mathbb{R}^s \times N^{n-s},\ \mathbb{R}^m = \mathbb{R}^s \times \mathbb{R}^{m-s}\ \text{and}\ f = I \times h,$$

where $h : N^{n-s} \to \mathbb{R}^{m-s}$ is an isometric immersion and $I : \mathbb{R}^s \to \mathbb{R}^s$ is the identity map. Clearly, in this case the minimal index of relative nullity ν_0 of f is precisely s, as long as that of h is zero.

The classical Hartman theorem states that these are the only possible complete examples with nonnegative Ricci curvature.

Theorem 1 (Maltz [1])**.** *Let M^n be a complete manifold with nonnegative Ricci curvature and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.*

The main purpose of this article is to extend the above result to submanifolds with Ricci curvature having a certain controlled decay.

Theorem 2. *Let M^n be a complete manifold with*

$$(1.1) \qquad \mathrm{Ric} \geq -\left(\mathrm{Hess}\,\psi + \frac{d\psi \otimes d\psi}{n-1} \right)$$

for some function ψ bounded from above on M^n and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.

Note that we recover Theorem 1 from the above by simply taking ψ to be constant.

Remarks 1. *(i) In Wylie* [2], *such a Riemannian manifold satisfying* (1.1) *was said to be* $CD(0, 1)$ *with respect to the* potential function ψ.
(ii) We actually prove a version of Theorem 2 that is more general in two ways. The first is that we can weaken the upper bound on ψ assumption to an integral condition along geodesics, the so-called bounded energy distortion. Secondly the function ψ can be replaced with a vector field X. We delay discussing this result until Section 4.

2. Preliminaries

The main step in the proof of Theorem 2 is Lemma 1 below (see Maltz [1]).

Lemma 1. *Suppose $M^n = \mathbb{R} \times N^{n-1}$ is the Riemannian product of $\mathbb{R}$ and a connected Riemannian manifold N^{n-1}, and suppose $f : M^n \to \mathbb{R}^m$ is an isometric immersion mapping a geodesic of the form $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$. Then f is a 1-cylinder.*

Our result also relies on the fundamental fact that the leaves of the minimum relative nullity distribution of a complete submanifold of $\mathbb{R}^m$ are also complete (cf. Dajczer [3]).

Lemma 2. *Let M^n be a complete Riemannian manifold and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with $\nu > 0$ everywhere. Then, the leaves of the relative nullity distribution are complete on the open subset where $\nu = \nu_0$ is minimal.*

Theorem 1 follows easily from Lemmas 1 and 2 above together with the Cheeger-Gromoll splitting theorem. Indeed, under the assumptions of Theorem 1, Lemma 2 yields that M^n contains ν_0 linearly independent lines through each point where the index of relative nullity is minimal. By the splitting theorem of Cheeger-Gromoll, M^n is isometric to a Riemannian product $\mathbb{R}^{\nu_0} \times N^{n-\nu_0}$, and Theorem 1 then follows inductively from Lemma 1.

The proof of our Theorem 2 uses the same ideas above, taking advantage of a recent warped product version of the splitting theorem by Wylie [2]. According to this latter result, estimate (1.1) is sufficient to split a complete Riemannian manifold M^n that admits a line into a warped product $\mathbb{R} \times_\rho N^{n-1}$ over $\mathbb{R}$. But since this splitting comes from a line of relative nullity, our goal is to show that the warping function ρ must be constant, and thus $\mathbb{R} \times_\rho N^{n-1}$ is actually a Riemannian product, so that Lemma 1 can be applied to conclude the proof. To do this we need to collect geometric information on the behavior of a warped product as above along the line $\mathbb{R}$. For later use, we carry out this study within the broader class of *twisted products* $M^n = \mathbb{R} \times_\rho N^{n-1}$ over $\mathbb{R}$, where (N, h) is a Riemannian manifold, $\rho : M^n \to \mathbb{R}_+$ the *twisting function*, and M^n is endowed with the metric $g = dr^2 + \rho^2 h$. If ρ is a function of r only, then we have a *warped product* over $\mathbb{R}$. The following lemma describes how vector fields vary along $\mathbb{R}$.

Lemma 3. *Let $M^n = \mathbb{R} \times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. Then*

$$\nabla_{\partial_r} \partial_r = 0 \tag{2.1}$$

and

$$\nabla_{\partial_r} X = \nabla_X \partial_r = \frac{1}{\rho} \frac{\partial \rho}{\partial r} X \tag{2.2}$$

for all $X \in \mathfrak{X}(N)$.

Proof. Let us write $\rho_r = \rho(r, \cdot)$ and denote by N_{ρ_r} the Riemannian manifold N endowed with the conformal metric rescaled by ρ_r^2. It is straightforward to check that ∇ given by (2.1), (2.2) and

$$\nabla_X Y = \nabla_X^{N_{\rho_r}} Y - \langle X, Y \rangle \frac{1}{\rho} \frac{\partial \rho}{\partial r} \partial_r$$

for all $X, Y \in \mathfrak{X}(N)$ defines a compatible symmetric connection on TM, hence it coincides with the Levi-Civita connection of M^n. $\qquad\square$

Next, we use Lemma 3 to compute the sectional curvatures along planes containing ∂_r.

Lemma 4. *Let $M^n = \mathbb{R} \times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. Then*

$$(2.3) \qquad\qquad K(\partial_r, X) = -\frac{1}{\rho} \frac{\partial^2 \rho}{\partial r^2}$$

for all unit vector $X \in T_x N$ and all $x \in N^{n-1}$.

Proof. Differentiating $\langle X, X \rangle = \rho^2$ twice with respect to r gives

$$\langle \nabla_{\partial_r} \nabla_{\partial_r} X, X \rangle + \|\nabla_{\partial_r} X\|^2 = \rho \frac{\partial^2 \rho}{\partial r^2} + \left(\frac{\partial \rho}{\partial r} \right)^2.$$

Using (2.1) and (2.2), we conclude that

$$\langle R(\partial_r, X) \partial_r, X \rangle = \rho \frac{\partial^2 \rho}{\partial r^2},$$

from which the result follows. $\qquad\square$

We are now in a position to state and prove our main lemma, in which by a *line of nullity* of a Riemannian manifold M^n we mean a curve $\gamma : \mathbb{R} \to M^n$ such that $\gamma'(t) \in \Gamma(\gamma(t))$ for all $t \in \mathbb{R}$, where

$$\Gamma(x) = \{X \in T_x M : R(X, Y) = 0 \text{ for all } Y \in T_x M\}$$

is the nullity subspace at $x \in M^n$.

Lemma 5. *Let $M^n = \mathbb{R} \times_\rho N^{n-1}$ be a twisted product over $\mathbb{R}$. If $\mathbb{R} \times \{q\}$ is a line of nullity of M^n for some $q \in N^{n-1}$, then $\rho_r = \rho_0$ does not depend on r, and hence M^n is actually the Riemannian product $\mathbb{R} \times N_{\rho_0}^{n-1}$.*

Proof. It follows from (2.3) that

$$\frac{\partial^2 \rho}{\partial r^2} \equiv 0,$$

but since the twisting function ρ is positive on the whole real line it must be constant. $\qquad\square$

3. Proof

As previously discussed, Lemma 1 is at the core of the proof of Theorem 2, whereas Lemma 5 is the principle behind its use.

Proof. We can assume that $\nu_0 = 1$, since the general case follows easily by induction on ν_0. Take a point $p \in M^n$ where $\nu = 1$. It follows from Lemma 2 that M^n contains a line l through p. By the warped product version of the splitting theorem of Cheeger-Gromoll due to Wylie [2], the Riemannian manifold M^n is isometric to a warped product $\mathbb{R} \times_\rho N^{n-1}$ over $\mathbb{R}$, the line l corresponding to $\mathbb{R} \times \{q\}$ for some $q \in N^{n-1}$. Since l is a leaf of the relative nullity foliation, we have in particular that $\mathbb{R} \times \{q\}$ is a line of nullity of $\mathbb{R} \times_\rho N^{n-1}$, and thus, by Lemma 5, $\rho_r = \rho_0$ does not depend on r and $\mathbb{R} \times_\rho N^{n-1}$ is actually the Riemannian product $\mathbb{R} \times N^{n-1}_{\rho_0}$. Hence, we may consider $f : \mathbb{R} \times N^{n-1}_{\rho_0} \to \mathbb{R}^m$, and as f maps $\mathbb{R} \times \{q\}$ onto a straight line in $\mathbb{R}^m$, the result then follows from Lemma 1. $\square$

4. GENERALIZATION

In this section we explain how the result above also has a version for non-gradient potential fields. Curvature inequality (1.1) has a natural extension to vector fields X and can be regarded as the special case where $X = \nabla\psi$.

Our result in the gradient case assumes boundness of the potential function ψ. While there is no potential function for a non-gradient field, we can still make sense of bounds by integrating X along geodesics. Let X be a vector field on a Riemannian manifold M^n. Let $\gamma : (a, b) \to M^n$ be a geodesic that is parametrized by arc-length. Define

$$\psi_\gamma (t) = \int_a^t \langle \gamma' (s), X (\gamma (s)) \rangle \, ds,$$

which is a real valued function on the interval (a, b) with the property that $\psi'_\gamma (t) = \langle \gamma' (t), X (\gamma (t)) \rangle$. When $X = \nabla\psi$ is a gradient field then $\psi_\gamma (t) = \psi (\gamma (t)) - \psi (\gamma (a))$, in the non-gradient case we think of ψ_γ as being the anti-derivative of X along the geodesic γ. We now recall the notion of 'bounded energy distortion', introduced by Wylie [2].

Definition 1. *Let M^n be a non-compact complete Riemannian manifold and $X \in \mathfrak{X} (M)$ a vector field. Then we say X has* bounded energy distortion *if, for every point $x \in M^n$,*

$$\limsup_{r \to \infty} \inf_{l(\gamma)=r} \left\{ \int_0^r e^{-\frac{2\psi_\gamma(\gamma(s))}{n-1}} ds \right\} = \infty,$$

where the infimum is taken over all minimizing unit speed geodesics γ with $\gamma (0) = x$.

In general, ψ_γ depends on the parametrization of γ only up to an additive constant, so the notion of bounded energy distortion does not depend on the parametrization of the geodesic. Also note that if a vector field X has the property that ψ_γ is bounded for all unit speed minimizing geodesics then it has bounded energy distortion. However, even in the gradient case, bounded energy distortion is a weaker condition than ψ bounded above.

Our most general cylindricity theorem is the following.

Theorem 3. *Let (M^n, g) be a complete manifold with*

$$(4.1) \qquad\qquad \mathrm{Ric} \geq - \left(\frac{1}{2} L_X g + \frac{X^\sharp \otimes X^\sharp}{n-1} \right)$$

for some vector field X with bounded energy distortion and let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with minimal index of relative nullity $\nu_0 > 0$. Then f is a ν_0-cylinder.

In particular, when $X = \nabla\psi$, we conclude that Theorem 2 still holds under the weaker condition that ψ has bounded energy distortion rather than being bounded from above.

By Wylie [2], inequality (4.1) allows to split M^n as a twisted product $\mathbb{R} \times_\rho N^{n-1}$ over $\mathbb{R}$, provided there is a line. But since Lemma 5 actually holds for twisted products, the proof of Theorem 3 then follows by the same arguments as in Section 3.

References

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